The Frege-Hilbert Controversy

Discussion about modern axiomatic methods in Mathematics

Michel van Garrel
Master in Mathematics
Project in Philosophy of Mathematics
Swiss Federal Institute of Technology Lausanne
<michel.vangarrel@epfl.ch>

Director: Philipp Keller
Department of Philosophy
University of Geneva
Switzerland

Lausanne, July 5, 2007

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1 Introduction

The publication of Hilbert's *Grundlagen der Geometrie*[1] in 1899 had a deep impact on theoretical Mathematics. By the example of Euclidean geometry Hilbert introduced a new way of setting the bases for a mathematical theory. He gave for the first time rigorous axioms for Euclidean geometry and he proved their consistency and independence[2]. Mathematically seen he gave a solution to the fundamental problem: *How can a mathematical theory be put upon solid bases?* This problem is extremely important. If the bases of a mathematical theory is "false"[3] the whole theory might be falsified.

Hilbert's answer to this problem is very simple. Recall that a mathematical theory is based upon axioms. Hilbert's answer is that every set of sentences can be chosen as axioms as long as they are consistent and independent with respect to the chosen set of axioms. Furthermore *primitives*[4] of the theory can be implicitly defined by axioms. The axioms describe how primitives behave and not what they are. They can be everything that satisfies the axioms. An example of a set of axioms is given in section 2.1.

We now explain the notions of consistence and independence of a set (or system) of axioms. Let $\mathfrak{A}$ be a system of axioms. A contradiction in $\mathfrak{A}$ is a proposition ϕ such that in $\mathfrak{A}$ it is provable that ϕ is true and also that ϕ is false. A system of axioms $\mathfrak{A}$ is consistent if it does not contain a contradiction. Consistence defines the "truth" of the system of axioms. A set of axioms $\mathfrak{A}$ is independent if for any two non-equal axioms A_1 and A_2 in $\mathfrak{A}$ the proposition A_1 *implies* A_2 is false. Independence guarantees the minimality of the axioms.

The axiomatic approach of Hilbert was very fruitful in the development of other mathematical theories as for instance the theory of groups. But can we philosophically accept such a free approach to the foundations of Mathematics? The notorious philosopher of logic Frege did not accept Hilbert's approach by free axioms and implicit definitions. He started a correspondence with Hilbert. We will present the axioms Hilbert give for Euclidean Geometry in section 2 and some parts of the critique Frege did in section 3. In section 4 we will discuss some disagreements between Frege and Hilbert.

A century after the correspondence between Frege and Hilbert the approach of Hilbert has been widely accepted among mathematicians. Still it is important to ask the question about the foundation of Mathematics and it is the aim of this project.

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[1] See [Hil].

[2] We explain the notions of consistence and independence of a set of axioms in the latter.

[3] It has to be said what false does mean in this context. Hilbert solves this problem with a weaker notion of what is false in the bases of a mathematical theory as is clear from what follows.

[4] For an example of primitives see section 2.1.

2 Hilbert's *Grundlagen der Geometrie*

We first present Hilbert's axioms and implicit definitions of Euclidean geometry. The complete list has 20 axioms. We only give the first 12 axioms. The complete list is given in section A. The axioms are taken from *www.wikipedia.org*, see [Wik].

2.1 Some axioms of Euclidean geometry

The undefined primitives are: *point, line, plane*. There are three primitive relations:

- *Betweenness*, a ternary relation linking points;
- *Containment*, three binary relations, one linking points and lines, one linking points and planes and one linking lines and planes;
- *Congruence*, two binary relations, one linking line segments and one linking angles, each denoted by an infix $\cong$.

Note that line segments (or shortly segments), angles and triangles may each be defined in terms of points and lines using the relations of betweenness and containment. A ray is the unique (by the axioms (1) and (2)) line that is extended by two points (or alternatively two segments). For two points A and B, AB denotes the segment or ray (or alternatively line) which is uniquely generated by the point A and B. Their existence and uniqueness is guaranteed by the axioms (1) and (2) and the construction (not given here) of a segment.

All points, lines and planes in each of the following axioms are distinct unless otherwise stated.

I Incidence

(1) Given any two points there exists a line containing both of them.

(2) Given any two points there exists no more than one line containing both points. I.e., the line described in (1) is unique.

(3) A line contains at least two points and given any line there exists at least one point not on it.

(4) Given any three points not contained in one line there exists a plane containing all three points. Every plane contains at least one point.

(5) Given any three points not contained in one line there exists only one plane containing all three points.

(6) If two points contained in a line m lie in some plane α then α contains every point in m.

(7) If the planes α and β both contain the point A then α and β both contain at least one other common point.

(8) There exist at least four points not all contained in the same plane.

II Order

(9) If a point B is between the points A and C then B is also between C and A and there exists a line containing the points A, B and C.

(10) Given two points A and C there exists a point B on the line AC such that C lies between A and B.

(11) Given any three points contained in one line one and only one of the three points is between the other two.

(12) *Axiom of Pasch.* Let A, B and C be three points not contained in one line. Let m be a line contained in the plane ABC but not containing any of A, B or C. If m contains a point on the segment AB then m also contains a point on the segment AC or on the segment BC.

It is easily seen that every axiom in the preceding list is chosen as minimal as possible, i. e. giving as few information as possible. Hilbert showed that his system of 20 axioms is consistent and independent.

2.2 Interpretation of the axioms of Euclidean geometry

In a first approach Hilbert's axioms look confusing. They do not seem to give a basis for Euclidean geometry. The most confusing fact is that Hilbert did not say what a point, for instance, is. He also did not say what the relation betweenness means. So, how is this system of axioms to be interpreted?

The answer is simple: Euclidean geometry is everything that satisfies this set of axioms. In such a everything a point and the relation betweenness will be defined. For instance the space in which we live, $\mathbb{R}^3$, is an Euclidean space. What we see as a point in $\mathbb{R}^3$ and the usual relation of betweenness will satisfy all the axioms.

It is also a simple and though deep interpretation of what axioms are. The notion of a point is defined implicitly.

3 Frege's critique to *Grundlagen der Geometrie*

The dissent between Frege and Hilbert is essentially due to a profoundly different understanding of how axiomatization should be handled. For Hilbert axioms can be chosen rather freely. Frege on the other hand thinks that every axiom should have a proper signification. We now give some points of the critique of Frege.

The basis of the critique of Frege can be summarized in the two questions[5]:

(a) *How can axioms be definitions?*

(b) *What sort of definitions are they if they fail to determine uniquely the terms they introduce?*

We choose two other points of Frege's critique, the first is an opposition to Hilbert and the second is a point of view of Frege.

(A) Frege thinks about axioms that they should express fundamental facts of our intuition[6]. They should be self-evident. Frege agrees to the need of introducing axioms in order to avoid circularity in proofs. But he rightfully says that Hilbert's axioms are not self-evident.

(B) According to Frege, it is of importance to give a meaning to primitive terms. To ensure that Frege uses the term *elucidation*. Elucidation should explain informally the primitive terms without defining them. Frege explains elucidation as follows[7]:

> *[Elucidation] is used to enable scientists to understand each other and to communicate science. It is to be counted as part of the propaedeutic. It has no place in the system of science; no inference is based upon it. Anyone who wishes to do research for and by himself has no need for it. The goal of elucidation is a practical one, and if it is attained, one must be satisfied with it. In doing this we must count on good will, halfway meetings of the understanding and hints; since without the use of metaphors we may never get anywhere. But it must be required of the elucidator that he knows definitely what he means, that he always agrees with himself, and that he is always ready, if the possibility of a misunderstanding arises (even when he is met with good will) to make his elucidation more complete and perfected. Since a collective enterprise in science is not possible without the mutual understanding of the researchers, we must trust that such an understanding can be reached by elucidation, although theoretically the contrary is non excluded.*

Frege wants to reduce Mathematics to pure logic, a project that had to be abandoned with Gödel's completeness theorems.

[5]See [Res], page 391.
[6]See [Res], page 388.
[7]See [Res], page 390.

4 Discussion

We analyze first the two questions (a) and (b). We start with question (a). Frege can not accept that a definition is given by one ore more axioms. In other words Frege does not accept implicit definitions. If one open a modern book of abstract Mathematics, there will be amounts of implicit definitions. Todays acceptance for implicit definition can be explained by its large success. It is a very fruitful approach in abstract Mathematics. And the application of abstract Mathematics to Physics, for instance, led to new precise theories as for example in quantum Physics or simply in the research of symmetries of the nature. These symmetries are given in terms of groups. Groups are defined implicitly. A posteriori it made sense to allow implicit definitions. We think that it is justified to accept implicit definitions. As a consequence the Philosophy of Mathematics is changed. This led later to the *Structuralism*[8].

Concerning question (b) we think that it is again a consequence to accept. It is also one of the very strong points of implicit definitions. Not the Euclidean geometry is defined but Euclidean geometry. One can do research in Euclidean geometry and compare it with other geometries. The examples of groups is more impressive. Since the definition of a group needs less axioms there are many "structures" that satisfy the definition of a group. One can start a classification of groups. Again, research done about groups had rich consequences in Physics for example. In our point of view it is profitable to accept that definitions do not necessarily define unique structures.

We consider now point (A). Frege relates axioms and intuition. It is necessary to build a mathematical theory from axioms. According to Frege this is easiest done by putting axioms in relation with intuition. But in this way Mathematics looses in freedom. And can we trust our intuition to be a good basis for Mathematics? We think not. What Hilbert does in some way is to free Mathematics from intuition. That makes Mathematics a discipline for itself, without the need of human interpretation. On the other hand, there is a loose of concrete understanding of Mathematics. We think that it is one of the essential characteristics of Mathematics to exist for itself and agree therefore with Hilbert.

Finally we discuss the elucidation Frege describes in point (B). In order to advance in science it is of interest for the scientists to be able to communicate science. But is this done by Frege's elucidation? We compare it with the way Mathematicians communicate each other orally their ideas. From a certain level on (typically after a PHD) Mathematicians start to talk in a very sophisticated way about Mathematics. They talk about actual developments as if they would talk about the weather. It is a very fruitful way to advance in Mathematics. But it normally does not start in the propaedeu-

[8]For a reference about Structuralism, see [Sha].

tic as Frege says. And contrarily to Frege, it has an important place in the system of science. Ideas are exchanged without the need to give every detail (which would need a lot more time). One need a high level in Mathematics to be capable of transferring mathematical developments to spoken language. In this the technical details are left beside. Only the important points and ideas of a proof, for instance, are taken out. This allows to advance very rapidly and is therefore essential in the practice of Mathematics. It is important for research in Mathematics and has not only a practical goal, contrarily to Frege. As a science of structures[9] mathematical object are structures. As this they get a signification. Proofs are essentially logical reasonings. Seen as this, talking about Mathematics is not just a intuitive way to communicate science, it is a very precise way to communicate. Most importantly, ideas are exchanged. Good ideas are the basis of Mathematics. They do push forward abstract Mathematics in particular.

5 Conclusion

Frege did the important task to ask questions about Hilbert's new axiomatization way. The questions Frege asks are pertinent and need an answer. In our opinion, we adopt a more free way of seeing Mathematics. Free in the sense that a set of axioms only need to be consistent and independent and in the sense that we allow implicit definitions. In accepting that we answer Frege's questions. The answer is roughly said "Yes we allow us to do that". Concerning elucidation we do not agree with Frege but suggest to see it differently, as part of the scientific developments.

In accepting implicit definitions and almost arbitrary sets of axioms we loose concreteness. But we gain in possibilities. Mathematics get freed from the concrete world and imagination gets its rightful place in Mathematics.

[9]For a book about structuralism, see [Sha].

A Hilbert's axioms of Euclidean geometry

We here give the complete list of the 21 axioms[10].

The undefined primitives are: *point*, *line*, *plane*. There are three primitive relations:

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[10] The axioms are taken from *www.wikipedia.org*, see [Wik].

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III Congruence

(13) Given two points A, B and a point A' on a line m there exist two and only two points C and D such that A' is between C and D and such that $AB \cong A'C$ and $AB \cong A'D$.

(14) If $CD \cong AB$ and $EF \cong AB$ then $CD \cong EF$.

(15) Let the line m include the segments AB and BC whose only common point is B. Let the line m or m' include the segments $A'B'$ and $B'C'$ whose only common point is B'. If $AB \cong A'B'$ and $BC \cong B'C'$ then $AC \cong A'C'$.

(16) Given the angle $\angle ABC$ and the ray $B'C'$ there exist two and only two rays $B'D$ and $B'E$ such that $\angle DB'C' \cong \angle ABC$ and such that $\angle EB'C' \cong \angle ABC$.

Corollary: Every angle is congruent to itself.

(17) Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.

IV Parallels

(18) *Playfair's postulate.* Given are a line m, a point A not on m and a plane containing both m and A. In that plane there is at most one line containing A and not containing any point on m.

V Continuity

(19) *Axiom of Archimedes.* Given the line segment CD and the ray AB there exist n points $A_1, \ldots, A_n$ on AB such that $A_jA_{j+1} \cong CD$, $1 \leq j < n$. Moreover, B is between A_1 and A_n.

(20) *Line completeness.* Adding points to a line results in an object that violates one or more of the axioms (1) - (14) and (19).

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