Grounding and Truth-functions

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Philosophers often make claims to the effect that certain facts obtain in virtue of, or are *grounded in*, other facts.[1] Among them one can find the following, or variants thereof:[2]

1. Mental facts obtain in virtue of neurophysiological facts;
2. Dispositional properties are grounded in categorical properties;
3. Legal facts are grounded in non-legal, e.g. social, facts;
4. Morally wrong acts are wrong in virtue of non-moral facts;
5. Normative facts are grounded in natural facts;
6. Semantic properties are exemplified in virtue of certain non-semantic properties being exemplified;
7. Determinables are exemplified in virtue of corresponding determinates being exemplified;
8. Universals exist in virtue of their having exemplifiers;
9. The existence of a whole is grounded in the existence of its parts;
10. Any given whole exists in virtue of the fact that its parts exist and are arranged in such and such a way;
11. The existence of a non-empty set is grounded in the existence of its members;
12. Events are grounded in facts about their participants;
13. Tropes are grounded in facts about their bearers;
14. Holes are grounded in facts about their hosts;
15. Every truth is made true, i.e. given any truth, some entity (or entities) is (are) such that that truth is true in virtue of the existence of this entity (these entities).

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[2] See e.g. Correia 2005 and Rosen 2010.

These claims and their negations constitute important philosophical theses, and, as the list indicates, talk of facts being grounded in other facts does not belong to a limited range of philosophical disciplines: it spreads over all areas of philosophy.

How is the relevant notion of grounding to be understood? A natural view on the issue invokes modality. The idea is that grounding should be characterized in terms of the notion of necessity, e.g. along the following lines: a fact is grounded in other facts iff (i) the latter facts obtain, and (ii) necessarily, if they obtain, then the former fact obtains as well. Yet a number of recent works convincingly questions the possibility of such modal renderings of the notion, and some authors even take the notion to be primitive, incapable of being characterized in fundamentally different terms.[3]

I am inclined to think that grounding cannot be understood in different terms. Whether this is right or wrong, however, I take it—certainly with many others—that a great deal of clarificatory work on the notion still needs to be done. This paper is intended to contribute to such a work, with a special emphasis on formal / logical issues pertaining to the notion of grounding. The focus of the paper is the propositional logic of grounding, i.e. the logic of the interaction between the notion of grounding and the truth-functions.[4]

The plan of the paper is the following. I first discuss the question of the logical form of statements of grounding (§1). There I distinguish between the predicational view on the logical form of these statements, and the operational view, which I endorse. I then introduce the notions of factual identity and factual equivalence, and argue that the formulation of a logic of grounding must go in tandem with the formulation of a logic of factual identity in case one opts for predicationalism, and of a logic of factual equivalence if one opts for operationalism (§2). In §3, I define the language relative to which I subsequently formulate the logic of grounding and factual equivalence. In §4 I lay down structural principles for grounding and factual equivalence. In §5, I then propose principles for the logic of factual equivalence and truth-functions, and in §6, I do the same for the logic of grounding and truth-functions. Finally, I present a semantical characterization of the resulting logical system and prove the system to be sound and complete with respect to the semantics (§7).

[3]See Audi ms, Correia 2005, Rosen 2010.

[4]Some clarificatory work on the notion of grounding, in particular on its logical properties, has been carried out by several authors, largely independently from each other. Bolzano 1973 is a notable precursor. In contemporary analytic philosophy, the notion has been largely ignored as a proper topic of philosophical inquiry until recent years. The most important works I am aware of are Audi ms, Correia 2005, Fine 2001, msa and msb, Rosen 2010, Schnieder 2004, 2006a, 2006b, 2006c, 2008 and ms. These works do not comprise properly logical studies of the sort I undertake in this paper, except for Fine msa and msb and Schnieder ms. These studies differ from the present one in important respects. In particular, Fine msa, contrary to Fine msb, does not deal with the interaction of grounding and truth-functions, and Schnieder ms studies only a binary notion of partial grounding, whereas I study a many-to-one notion of total grounding. I will point to other differences and to similarities between our respective works in due course.

1 Logical Form

In this section I first distinguish between the predicational and the operational views about the logical form of statements of grounding and advocate the latter view (§1.1), and I then argue that grounding is "many-to-one" (§1.2).

1.1 Predicational vs. Operational

Grounding claims usually take one of the following grammatical forms:

(1) The fact that p is grounded in the fact that q, the fact that r, ...;
(2) p in virtue of the fact that q, the fact that r, ...;
(3) p because q, r, ...;
(4) The fact that p is explained by the fact that q, the fact that r, ...

(1)-(4) have various apparent logical forms. Granted that (1)-(4) can be used interchangeably to say the very same thing, this variety must be only apparent, and so arises the question which underlying logical form we should take grounding claims to have.

There are two natural views on that matter. On one view, the most basic notion of grounding is to be expressed by means of a relational predicate, say 'IS GROUNDED IN', which takes designators for facts to make sentences. That view naturally comes to mind if we draw our attention to the surface grammar of (1) or (4). On the other view, the most basic notion of grounding is to be expressed by means of a sentential operator, say 'BECAUSE', which, like e.g. the truth-functional connectives 'and' and 'not', takes sentences to make a sentence. That view naturally comes to mind if we focus on the surface grammar of (3). Call the first view *predicational* and the second *operational*.[5]

I took care to define these two views as views concerning the logical category the *most basic* notion of grounding belongs to. This is because once a certain notion of grounding belonging to a logical category is given, it is possible, granted certain assumptions about the ontology of facts and the expressivity of our language for facts, to define another notion of grounding belonging to a different logical category. Thus, assuming the predicational view, we can define a sentential operator 'bec' as follows:

- p bec q, r, ... iff$_{df}$ the fact that p IS GROUNDED IN the fact that q, the fact that r, ...

And assuming the operational view, granted obvious assumptions about the ontology of facts about the expressivity of our language for facts, we can define a predicate 'is gred in' on facts as follows:

[5]One can think of a third view on which the most basic notion of grounding is to be expressed by means of a hybrid expression as suggested by (2). That view strikes me as very unnatural, and so I will leave it aside. Yet the material of this paper could easily be adapted to take it into account.

- The fact that p is gred in the fact that q, the fact that r, ... iff$_{df}$ p BECAUSE q, r, ...

Which conception of the most basic notion of grounding should be adopted?[6] I will go for the operational approach. My preference goes to that approach for reasons of ontological neutrality: it should be possible to make claims of grounding and fail to believe in facts.[7] Yet, for the sake of linguistic convenience I will feel free to read statements of type 'p BECAUSE q, r, ...' as 'the fact that p is grounded in the fact that q, the fact that r, ...', and to use fact-talk in other contexts where such talk is not intended to be understood literally. Such an informal talk of facts should not result in misunderstanding.

1.2 Grounding as Many-to-one

As was perhaps clear to the reader right from the start, I take grounding to be essentially many-to-one (and being grounded in to be one-to-many). Assuming the predicational conception of grounding, the view is that it may be true that:

(5) x IS GROUNDED IN y, z, ...

without it being true that:

- (x IS GROUNDED IN y) or (x IS GROUNDED IN z) or ....

To put it differently, the view is that some facts may jointly GROUND a fact without any one of the former facts, taken in isolation, GROUNDING the latter fact. Assuming the operational conception, the view is that:

(6) p BECAUSE q, r, ...

may be true without:

- (p BECAUSE q) or (p BECAUSE r) or ...

being true.

It may be thought that, despite this feature, every case of grounding reduces to a case of one-one grounding. The view I have in mind, applied to the predicational conception, is that (5) should be understood as saying that x IS GROUNDED IN a "conjunctive fact" whose conjuncts are y, z, ...; and applied to the operational conception, the view is that (6) should be understood as saying that p BECAUSE (q and r and ...). Yet, both reductions are questionable. For

[6] Audi ms and Rosen 2010 express grounding by means of a predicate, Correia 2005, Fine msa, msb and Schnieder ms (and elsewhere) by means of a sentential operator. Yet they do so without claiming that the expressions they use are intended to point to the most basic notion. Mulligan 2004 goes for the operational view.

[7] As far as I can see, there are two main objections to admitting facts in one's ontology. One is nominalistic in character: accepting the existence of facts is accepting the existence of abstract entities, but there are no such entities. The other one is that accepting an ontology of facts amounts to a mistaken projection of features of language onto reality. Of course, the neutrality I aim for would equally be secured by going predicationalist and having an appropriately deflationary conception of facts.

while it is plausible to say that at least some instances of ‘the fact that p and q IS GROUNDED IN the fact that p and the fact that q’ or of ‘p and q BECAUSE p, q’ are true, no instance of ‘the fact that p and q IS GROUNDED IN the fact that p and q’ or of ‘p and q BECAUSE p and q’ can be true: grounding is, I take it, irreflexive.[8] I shall accordingly treat grounding as many-to-one.[9]

2 Grounding and Factual Equivalence

Take for a moment the predicational view about grounding for granted. Then the following two general principles connecting grounding and identity between facts should be countenanced:

- For all facts x, y and z, if both $x = y$ and z IS GROUNDED IN x, then z IS GROUNDED IN y;
- For all facts x, y and z, if both $x = y$ and x IS GROUNDED IN z, then y IS GROUNDED IN z.

They are indeed just instances of the Principle of the Indiscernibility of Identicals for the predicate ‘IS GROUNDED IN’, and to that extent they are uncontroversial.

Let me henceforth use ‘$[p]$’ for ‘the fact that p’. With these principles in place, the following schemas should be taken to have all their instances true:

- If both $[p] = [q]$ and $[r]$ IS GROUNDED IN $[p]$, then $[r]$ IS GROUNDED IN $[q]$;
- If both $[p] = [q]$ and $[p]$ IS GROUNDED IN $[r]$, then $[q]$ IS GROUNDED IN $[r]$.

Thus there naturally arises, in the context of the predicational view about grounding, the question of when statements of type ‘$[p] = [q]$’ should be taken to be true. Given that the canonical way we have at our disposal for referring to facts, if there are such things, is by means of expressions of type ‘the fact that p’, that question is not a mere side question for the predicationalist: it is absolutely central.

In the context of the predicational view of grounding, a distinction is to be made between two classes of conceptions of the nature of facts, the conceptions of facts as “worldly” and the conceptions of facts as “conceptual”, which makes a huge difference on the assessment of statements of type ‘$[p] = [q]$’.[10] On a

[8] Most if not all contemporary authors agree on this point. There is a well known philosophical tradition which takes it that there are cases of facts grounded in themselves, most notoriously facts about God.

[9] Most authors do the same. As I previously emphasized, Schnieder ms studies a binary notion of partial grounding. It may be thought that his notion can be defined in terms of a many-to-one notion along the following lines: a fact is grounded in another fact (binary notion) iff$_{df}$ the former fact is grounded (many-to-one notion) in the latter fact, or in the latter fact together with other facts.

[10] The distinction appears in Fine msb, where Fine introduces his semantics for the logic of grounding.

conception of facts as conceptual, which fact is referred to by means of a description of type '[p]' is highly sensitive to the concepts involved in that description; on a conception of facts as worldly, there is significantly less sensitivity. As an illustration of what I have in mind, consider the following three lists of facts (a is any arbitrary water molecule):

- [a is a water molecule]; [a is an H_2O molecule];
- [France is east of Argentina]; [Argentina is west of France];
- [France is east of Argentina]; [it is not the case that it is not the case that France is east of Argentina].

It will naturally be held, on a conception of facts as worldly, that each list is constituted by one fact appearing twice. In contrast, on a conception of facts as conceptual, each list will naturally be taken to comprise two distinct facts. (Thus, on a conception of facts as conceptual, it is probably more appropriate to call facts 'propositions' rather than 'facts'.)

Let us now turn to the operational view. Here, of course, the corresponding issue about the identity of facts does not arise—at any rate, not immediately. Yet it is my view that something very much like it does arise.

For suppose we are operationalists and want to say e.g. of a certain instance of 'p BECAUSE Sam and Maria like each other' that it is true. Then we will surely have to say that the corresponding instance of 'p BECAUSE Maria and Sam like each other' is true as well. Why is that? The natural thing to say here is that there is a sense of 'saying the same thing' such that, understood in that sense we must concede that (i) 'Sam and Maria like each other' and 'Maria and Sam like each other' say the same thing, and (ii) whenever 'q' and 'r' say the same thing, 'p BECAUSE q' and 'p BECAUSE r' are bound to have the same truth-value. Of course, similar considerations hold if we start from a given instance of 'Sam and Maria like each other BECAUSE p' instead. Thus for the operationalist, there will be a notion of 'saying the same thing' for which he will take all instances of following schemas to be true:

- If both 'p' and 'q' say the same thing and 'r BECAUSE p' is true, then 'r BECAUSE q' is true;
- If both 'p' and 'q' say the same thing and 'p BECAUSE r' is true, then 'q BECAUSE r' is true.

Now this being said, there arises the question which statements say the same thing, in the relevant sense, and which ones do not. For instance, we may ask:

- Do 'a is a water molecule' and 'a is an H_2O molecule' say the same thing?
- Do 'France is east of Argentina' and 'Argentina is west of France' say the same thing?
- Do 'France is east of Argentina' and 'it is not the case that it is not the case that France is east of Argentina' say the same thing?

There is room for disagreement. Some operationalists will have a "worldly" conception of 'saying the same thing', and will in particular naturally take the answer to these questions to be positive, while others will have a "conceptual" conception of 'saying the same thing', and will in particular naturally answer these questions negatively.

Thus, corresponding to the issue for predicationalists of how statements of type:

(7) $[p] = [q]$

should be assessed, there is the issue for operationalists of how statements of type:

- 'p' and 'q' say the same thing

should be assessed, and the latter issue is obviously as important for the operationalists as the former issue is for the predicationalists.

The notion of saying the same thing has been expressed up to now by means of a binary predicate on sentences. For reasons of formal homogeneity and convenience I will henceforth take it to be canonically expressed by means of a sentential operator '$\approx$' dubbed 'factual equivalence'. Thus a statement of type:

(8) $p \approx q$

will be taken to be true iff the corresponding statements 'p' and 'q' say the same thing. (8) can itself be read 'its being the case that p and its being the case that q are one and the same thing', or even, informally of course, as 'the fact that p and the fact that q are one and the same fact'. I will call statements of type (7) statements of *factual identity* and statements of type (8) statements of *factual equivalence*. Given my operationalist stance, I will mainly focus on factual equivalence as opposed to factual identity.

Many alternative conceptions of facts as worldly can certainly be put forward in the context of a theory of grounding, and the same is true of conceptions of facts as conceptual. The same holds, *mutatis mutandis*, for worldly and conceptual conceptions of factual equivalence. I will largely remain silent on which conceptions are available and on which are better than others—given the scope of this paper there is no need to say much on these matters.

I will nevertheless adopt a worldly conception of factual equivalence. In fact, the conceptualist has a very fine-grained conception of factual equivalence, and I believe that this conception is too fine-grained. I take, along with others, grounding to "carve reality at the joints". Now the problem with conceptualism is that, in any of its plausible versions, it yields true statements of grounding which, intuitively, do not correspond to the way reality is carved up. Assume for instance with the conceptualist that the two sentences 'a is a water molecule' and 'a is an H_2O molecule' are not factually equivalent (change the example if you do not find this one appropriate). Then, plausibly, '(a is both a water molecule and an H_2O molecule) BECAUSE a is a water molecule, a is an H_2O molecule' is true. Yet it is hard to see how it could be argued that this truth depicts some aspect of the structure of reality. Or again, granted that 'a is a water molecule' and 'a is an H_2O molecule' are not factually equivalent, it is plausible to hold that '(a is a water molecule or 2+2 = 4 BECAUSE a is a water molecule' and 'a is an H_2O molecule or 2+2 = 4 BECAUSE a is an H_2O molecule' are both true, and express different truths. Yet one feels that this difference does not correspond to a relevant metaphysical distinction.[11]

3 The Formal Language and the Basic Logical Machinery

The language, L, relative to which I will formulate a logic of grounding and factual equivalence is a propositional language with $\wedge$ (conjunction) $\neg$ (negation) and $\vee$ (disjunction) as primitive truth-functional connectives. Material implication and material equivalence are defined in terms of these connectives in the usual way: $A \supset B$ is defined as $\neg A \vee B$ and $A \equiv B$ as $(\neg A \vee B) \wedge (\neg B \vee A)$. L contains the sentential operators $\approx$ for factual equivalence and $\mathcal{B}$ for grounding. The atoms of L are of two sorts: they divide into the *sentential variables* x, y, ..., which we assume to be ordered, and the *sentential constants* a, b, ... Language L contains the quantifier $\exists$. Finally, the brackets (and) belong to the vocabulary of L, as well as the list builder ,.

The *basic formulas* of L are defined recursively as follows:

- The atoms are basic formulas;
- If p and q are basic formulas, then so are $(p \wedge q)$, $(\neg p)$ and $(p \vee q)$.

A *list* is a sequence of one or several basic formulas separated by ,. The *formulas* of L are defined recursively as follows:

- The basic formulas are formulas;
- If p and q are basic formulas, then $(p \approx q)$ is a formula;
- If p is a basic formula and Δ a list, then $(p \mathcal{B} \Delta)$ is a formula;
- If A and B are formulas, then so are $(A \wedge B)$, $(\neg A)$ and $(A \vee B)$;
- If A is a formula and x a sentential variable, then $(\exists x A)$ is a formula.

I will use:

- p, q, ... for arbitrary basic formulas;
- A, B, ... for arbitrary formulas
- Δ, Γ, ... for arbitrary lists;
- $\widehat{\Delta}$ for p_1 if Δ is p_1, for $(p_1 \wedge p_2)$ if Δ is p_1, p_2, for $((p_1 \wedge p_2) \wedge p_3)$ if Δ is p_1, p_2, p_3, etc.

[11]The importance of the notion of factual equivalence or factual identity in the context of a theory of grounding has not been properly appreciated. Audi ms and Rosen 2010 both take grounding to be a relation between facts, and both elaborate a bit on the nature of these facts, but not much. Audi advocates a worldly conception of facts, Rosen a conceptual conception. It should be noted that in addition to the notion of grounding, which he dubs "strict", Fine (msa and msb) introduces a notion of "weak" grounding, which plays a somewhat similar formal role as factual equivalence. I will comment on Fine's notion a bit later.

I will follow standard notational conventions. I define:

- $p \geqslant^d q$ as $\exists x(p \approx (q \vee x))$

and:

- $p \geqslant^{cd} q$ as $\exists y(p \geqslant^d (q \wedge y))$

(x the first variable in the assumed ordering not in p or q, and y the first variable in the assumed ordering not in $p \geqslant^d q$). We may read $p \geqslant^d q$ as 'the fact that p disjunctively contains the fact that q', and $p \geqslant^{cd} q$ as 'the fact that p disjunctively contains some fact which conjunctively contains q'. Finally, I will use:

- $p \not\approx q$ for $\neg(p \approx q)$,
- $p \not\geqslant^d q$ for $\neg(p \geqslant^d q)$

and:

- $p \not\geqslant^{cd} q$ for $\neg(p \geqslant^{cd} q)$.

The logic for language L, which I will call 'G', will be formulated in a Hilbert-style format. I will assume all L-instances of classical tautologies as axioms of G, Modus Ponens as a rule of the system, and finally I will assume that standard postulates for existential quantification are also part of G.

Some remarks about the choice of that language are in order.

(1) The language is "first degree": it does not allow for the occurrence of $\approx$ or $\mathcal{B}$ within the scope of an occurrence of $\approx$ or $\mathcal{B}$. This is mainly due to the fact that it is hard, except in special cases, to evaluate claims involving corresponding embeddings, like e.g. many claims of the form '(p BECAUSE q) BECAUSE r'.

(2) The language does not allow for the presence of the quantifier within the scope of $\approx$ or $\mathcal{B}$. This is due to the fact that the object of this paper is the study of the interaction between grounding and the truth-functions, not the study of its interaction with truth-functions and quantification.

(3) The language only allows for finite lists, whereas it would have been more natural to allow for lists of arbitrary sizes. That choice was guided by considerations of simplicity.

(4) The language is quantified, whereas one would have thought that a purely propositional language could have been appropriate. The reason why I introduced quantification is that $\geqslant^d$ and $\geqslant^{cd}$, which have been defined in terms of $\exists$, play a crucial role in the logic I will introduce. Anticipating a little bit (see §5), I should say that thanks to the specific logic of factual equivalence I will advocate, it turns out that $\geqslant^d$ and $\geqslant^{cd}$ are definable without the help of the quantifier, and so a purely propositional logic of grounding and factual equivalence pretty much like G can be defined, and semantically characterized in essentially the same way as G (see the last part of §7). Yet I shall focus on the quantified language, for reasons of naturalness, and also in order to facilitate

the comparison between my logic and the logic of others for whom $\geqslant^d$ and $\geqslant^{cd}$ cannot be defined without the help of $\exists$.

(5) The language does not allow for plural sentential quantifiers, whereas one would have thought that, due to the many-to-one character of grounding, allowing for plural quantification would have been more natural. There are two reasons why I left plural quantification aside. The first is that while singular sentential quantification is quite controversial, plural sentential quantification is certainly much more controversial. The second reason is that with plural quantification (standardly interpreted) in place, the prospects of getting a completeness result of the sort I establish in §7 are null.

4 Structural Principles

Which structural principles, i.e. principles which do not concern the interaction with truth-functions, do factual equivalence and grounding conform to? I suggest that all the L-instances of the following schemas involving $\approx$ should be amongst the theorems of G:

E1 $p \approx p$ *Reflexivity*
E2 $(p \approx q) \supset (q \approx p)$ *Symmetry*
E3 $(p \approx q) \wedge (q \approx r) \supset (p \approx r)$ *Transitivity*
E4 $(p \approx q) \supset (p \equiv q)$. *Quasi-factivity*

as well as all the L-instances of the following formulas involving $\mathcal{B}$:

P $(p\,\mathcal{B}\,\Delta) \supset (p\,\mathcal{B}\,\Gamma)$ — Γ a reordering of Δ *Permutation*
R $(p\,\mathcal{B}\,\Delta) \equiv (p\,\mathcal{B}\,\Delta, q)$ — q any item in Δ *Repetition*
S1 $(p\,\mathcal{B}\,\Delta) \wedge (q \approx p) \supset (q\,\mathcal{B}\,\Delta)$ *L-substitution*
S2 $(p\,\mathcal{B}\,\Delta, q) \wedge (q \approx r) \supset (p\,\mathcal{B}\,\Delta, r)$ *R-substitution*
G1 $(p\,\mathcal{B}\,\Delta) \supset (p \wedge \widehat{\Delta})$ *Factivity*
G2 $\neg(p\,\mathcal{B}\,\Delta, p)$ *Irreflexivity*
G3 $(p\,\mathcal{B}\,\Delta, q) \wedge (q\,\mathcal{B}\,\Gamma) \supset (p\,\mathcal{B}\,\Delta, \Gamma)$ *Cut*

That all the L-instances of **E1-E4**, **P**, **R**, **S1** and **S2** should count as theorems of G hardly needs to be justified; and there would seem to be an almost universal consensus that all the L-instances of **G1-G3** should also count as such.[12]

Granted that all the L-instances of the highlighted schemas are theorems of G, it can be shown that the same goes for all the L-instances of the following schemas:

- $(p\,\mathcal{B}\,q) \wedge (q \approx r) \supset (p\,\mathcal{B}\,r)$ *R-substitution**
- $(p\,\mathcal{B}\,\Delta) \supset (q \not\approx p)$ — q any item in Δ *Irreflexivity**

[12]See Audi ms, Correia 2005, Fine msa, msb, Rosen 2010, and Schnieder ms for statements / arguments which go in the direction of having some or all of **G1-G3**. Fine msa does not have a postulate corresponding to Factivity, but interprets the formulas which flank his grounding operators as truths.

- $\neg(p\,\mathcal{B}\,p)$ *Irreflexivity***
- $(p\,\mathcal{B}\,q) \wedge (q\,\mathcal{B}\,\Gamma) \supset (p\,\mathcal{B}\,\Gamma)$ *Transitivity*
- $(p\,\mathcal{B}\,\Delta, q) \supset \neg(q\,\mathcal{B}\,\Gamma, p)$ *Anti-circularity*
- $(p\,\mathcal{B}\,q) \supset \neg(q\,\mathcal{B}\,p)$ *Asymmetry*

I shall take the highlighted schemas for $\mathcal{B}$ as axiom schemas for system G, as well as **E2-E4** (but under a different name). I will not take **E1** as an axiom schema, though, because it is derivable from further axiom schemas for factual equivalence I will adopt (see next section).

5 Factual Equivalence and Truth-functions

Which postulates for the interaction between $\approx$ and the truth-functions should we take for granted?

Let us tackle the question from an informal point of view and let us first think about sufficient conditions for factual equivalence. There are three candidates I wish to mention first—they are all inadequate, but it will be instructive to see why:

(a) If 'p' and 'q' are materially equivalent, then 'p' and 'q' are factually equivalent;
(b) If 'p' and 'q' are metaphysically necessarily equivalent, then 'p' and 'q' are factually equivalent;
(c) If 'p' and 'q' are logically equivalent according to classical logic, then 'p' and 'q' are factually equivalent.

(Classical logical equivalence is understood here in the sense of the classical propositional calculus.)

Given previously accepted principles, each of the proposed sufficient conditions is too weak. Given that grounding is factive and irreflexive, and obeys the substitution principles formalized as **S1** and **S2** in system G, a consequence of (a) is that no instance of 'p BECAUSE q' can be true. For suppose 'p BECAUSE q' is true. Then by factivity, both 'p' and 'q' are true, and hence materially equivalent. By (a) it follows that 'p' and 'q' are factually equivalent, and by a substitution principle that 'p BECAUSE p'—which goes against irreflexivity.

That the condition in (c)—and consequently, the condition in (b)—is too weak as well can be argued for on two grounds. The first has to do with relevance. Suppose an instance of 'p BECAUSE q' is true. Given that 'q' and '$q \wedge (p \vee \neg p)$' are logically equivalent according to classical logic, by either (b) or (c) they are factually equivalent, and so by substitution 'p BECAUSE $q \wedge (p \vee \neg p)$' should be true. Yet, this will be denied, on the grounds that (to use the predicationalist idiom) the fact that $p \vee \neg p$ can play absolutely no role to help ground the fact that p. The other argument does not involve relevance, at any rate not in the same way. I take it that on a worldly conception of factual equivalence, some instances of '$q \vee (q \wedge p)$' are not factually equivalent to the corresponding instances of 'q', and that whenever '$q \vee (q \wedge p)$' is not factually equivalent to 'q' and 'q' is true, '$q \vee (q \wedge p)$ BECAUSE q' should be taken to be true. Now '$q \vee (q \wedge p)$' and 'q' are classically equivalent, and consequently both (b) and (c) conflict with the worldly conception.[13]

Interestingly, the second argument against (c) excludes a further suggestion, which invokes *relevant equivalence* in the sense of the system of first degree entailment of Anderson and Belnap (1975):

(d) If 'p' and 'q' are relevantly equivalent, then 'p' and 'q' are factually equivalent.

The suggestion may appear *prima facie* plausible, since logical relevant equivalence is significantly stronger than classical logical equivalence. But it must be rejected, since 'q' and '$q \vee (q \wedge p)$' are not only logically equivalent according to classical logic, they are also relevantly so.

I wish to suggest that the *logic* of factual equivalence, taken on a worldly conception, is the logic of analytic equivalence in the sense of R. B. Angell (see Angell 1977, 1989 and Correia 2004).[14] Relative to a standard propositional language with conjunction, negation and disjunction as primitive truth-functional connectives, enriched with the binary sentential operator $\leftrightarrow$ for analytic equivalence, a Hilbert-style axiomatization of the notion is given by the following axiom schemas and rule of inference:[15]

ae0 All classical tautologies
ae1 $p \leftrightarrow \neg\neg p$
ae2 $p \leftrightarrow p \wedge p$
ae3 $p \wedge q \leftrightarrow q \wedge p$
ae4 $p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$
ae5 $p \vee q \leftrightarrow \neg(\neg p \wedge \neg q)$
ae6 $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$
ae7 $(p \leftrightarrow q) \supset (\neg p \leftrightarrow \neg q)$
ae8 $(p \leftrightarrow q) \supset (p \wedge r \leftrightarrow q \wedge r)$
ae9 $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \supset (p \leftrightarrow r)$
ae10 $(p \leftrightarrow q) \supset (q \leftrightarrow p)$
ae11 $(p \leftrightarrow q) \supset (p \equiv q)$
ae12 $p, p \supset q \,/\, q$

[13]Fine convinced me that a previous version of the second argument was not effective.

[14]I emphasize that I take the *logic* of the two notions to be the same, because I do not think the notions are the same. Angell takes analytic equivalence to boil down to a form of sameness of meaning (1989, p. 119), and I certainly do not want to treat factual equivalence that way. My work on Angell's logic, which does not mention the concept of grounding, derived in fact form my work on the concept: certain properties of the logic made me think analytic equivalence or some notion in the vicinity could be helpful to model grounding.

[15]See Angell 1977. Angell's system is formulated only with negation and conjunction as primitive, and he defines disjunction in the usual way in terms of these concepts. Angell thus does not have axiom schema ae5. And instead of ae11 he has (equivalently given the rest of the system) $(p \leftrightarrow p \wedge q) \supset (p \supset q)$.

(Notice that $p \leftrightarrow p$ follows from ae1, ae9 and ae10, and so need not be adopted as a further postulate.) I will thus take the result of replacing $\leftrightarrow$ by $\approx$ in ae1-ae11 as further axiom schemas for system G (ae0 and ae12 are already present).

The objections given above against (a)-(d) do not affect the proposed conception of factual equivalence. In fact, the resulting logic of factual equivalence validates neither of the following relevant schemas:

- $(p \equiv q) \supset (p \approx q)$;
- $q \approx (q \wedge (p \vee \neg p))$;
- $q \approx (q \vee (q \wedge p))$.

For the record, an axiomatization of relevant equivalence can be obtained from ae0-ae12 by adding the axiom schema $q \leftrightarrow q \vee (q \wedge p)$, and by adding to the latter system the axiom schema $q \leftrightarrow q \wedge (p \vee \neg p)$, $\leftrightarrow$ gets to behave just like material equivalence.

Granted the proposed logic of factual equivalence, the following are theorems of system G (the definitions of $\geqslant^d$ and $\geqslant^{cd}$ have been given in §3):

- $p \geqslant^d p$
- $(p \geqslant^d q) \wedge (q \geqslant^d p) \supset p \approx q$
- $(p \geqslant^d q) \wedge (q \geqslant^d r) \supset (p \geqslant^d r)$
- $(p \geqslant^d q) \supset (p \wedge r \geqslant^d q \wedge r)$
- $(p \geqslant^d q) \supset (p \vee r \geqslant^d q \vee r)$
- $(p \geqslant^d q) \supset (p \vee r \geqslant^d q)$
- $(p \geqslant^d q) \supset (p \geqslant^d p \wedge q)$
- $(p \geqslant^d q) \supset (p \geqslant^d p \vee q)$
- $(p \geqslant^d q) \supset (q \supset p)$

- $p \geqslant^{cd} p$
- $(p \geqslant^{cd} q) \wedge (q \geqslant^{cd} r) \supset (p \geqslant^{cd} r)$
- $(p \geqslant^{cd} q) \supset (p \wedge r \geqslant^{cd} q)$
- $(p \geqslant^{cd} q \wedge r) \supset (p \geqslant^{cd} q)$
- $(p \geqslant^{cd} q \vee r) \supset (p \geqslant^{cd} q)$
- $(p \geqslant^{cd} q) \supset (p \vee r \geqslant^{cd} q)$

- $(p \geqslant^d q) \supset (p \geqslant^{cd} q)$
- $(p \geqslant^d q) \wedge (q \geqslant^{cd} r) \supset (p \geqslant^{cd} r)$
- $(p \geqslant^{cd} q) \wedge (q \geqslant^d r) \supset (p \geqslant^{cd} r)$

Importantly, the following equivalences, which show that we could have defined $\geqslant^d$ and $\geqslant^{cd}$ without the resources of existential quantification, also hold:

d1 $(p \geqslant^d q) \equiv (p \approx q \vee p)$
d2 $(p \geqslant^{cd} q) \equiv (p \geqslant^d q \wedge p)$

The right-to-left direction of each equivalence directly follows from quantification theory. For the other directions, here are informal arguments. (Here and below, when I present informal arguments there is always a corresponding formal argument.) For d1, suppose $p \geqslant^d q$, and let x be such that $p \approx q \vee x$. Then $p \vee q \approx (q \vee x) \vee q$, and so $p \vee q \approx q \vee x$. But then by the initial assumption, $p \approx q \vee p$. (2) For d2, suppose $p \geqslant^{cd} q$, and let y be such that $p \geqslant^d q \wedge y$. By d1, $p \approx (q \wedge y) \vee p$. It follows that $q \wedge p \approx q \wedge ((q \wedge y) \vee p)$. By a distributivity property, we then have $q \wedge p \approx (q \wedge (q \wedge y)) \vee (q \wedge p)$, and so $q \wedge p \approx (q \wedge y) \vee (q \wedge p)$. By a distributivity property, we have then $q \wedge p \approx q \wedge (y \vee p)$. It follows that $(q \wedge p) \vee p \approx (q \wedge (y \vee p)) \vee p$. By a distributivity property, it follows that $(q \wedge p) \vee p \approx (q \vee p) \wedge ((y \vee p) \vee p)$, and so $(q \wedge p) \vee p \approx (q \vee p) \wedge (y \vee p)$. By a distributivity property, it follows that $(q \wedge p) \vee p \approx (q \wedge y) \vee p$. But then by the initial hypothesis, $p \approx (q \wedge p) \vee p$, and consequently, $p \geqslant^d q \wedge p$.

So far for the logic of factual equivalence. It has been formulated on the assumption of a worldly conception of the notion, which I endorsed from the beginning. Yet a glimpse at conceptual views on factual equivalence may be useful for clarificatory purposes. How, then, is the conceptualist to characterize the logic of factual equivalence?

I have no clear view on these matters, only some suggestions. I suggest a conceptualist may be happy to accept the following schemas, which are all validated by my logic of worldly factual equivalence:

- $p \wedge q \approx q \wedge p$
- $p \wedge (q \wedge r) \approx (p \wedge q) \wedge r$
- $p \vee q \approx q \vee p$
- $p \vee (q \vee r) \approx (p \vee q) \vee r$
- $(p \approx q) \equiv (\neg p \approx \neg q)$
- $(p \approx q) \supset (p \wedge r \approx q \wedge r)$
- $(p \approx q) \supset (p \vee r \approx q \vee r)$
- $(p \approx q) \wedge (q \approx r) \supset (p \approx r)$
- $(p \approx q) \supset (q \approx p)$
- $(p \approx q) \supset (p \equiv q)$

A guiding view for the conceptualist might be that two statements containing different connectives cannot be factually equivalent, as well as statements containing the same connectives but occurring a different number of times. A principle which may be added to the previous postulates which conforms to that view is:

- $p \not\approx q$ if q strictly contains p (i.e. contains, but is not identical to, p).

Of course, with that addition the resulting system is no longer a fragment of the worldly system.

A stronger guiding view for a conceptualist might be the following: two statements are factually equivalent iff they have just the same structure and contain equivalent atoms at the same points in the structure. The newly added postulate would still hold on that view, but the first four postulates of the previous list would have to be dropped. The following principles should then be added:

- $(p \wedge r \approx q \wedge r) \supset (p \approx q)$
- $(p \vee r \approx q \vee r) \supset (p \approx q)$

Notice that neither of these principles hold in the worldly system. Take the first one for instance. In the worldly system, $(q \wedge r) \wedge (q \wedge r) \approx q \wedge (q \wedge r)$ is a theorem, and yet $(q \wedge r) \approx q$ is not. Also notice that for a conceptualist, $p \not\approx q \vee p$ and $p \not\geqslant^{d} q \wedge p$ must hold, and so he cannot accept d1 if he grants that some instances of $p \geqslant^{d} q$ can be true, and d2 if he grants that some instances of $p \geqslant^{cd} q$ can be true.

6 Grounding and Truth-functions

Let us finally turn to the question which postulates for the interaction between $\mathcal{B}$ and the truth-functions should be accepted. For the sake of readability, in this section I shall use:

- $\frac{A}{A_0}$ for $A \supset A_0$
- $\frac{A\,B}{A_0}$ for $(A \wedge B) \supset A_0$
- $\frac{A\,B\,C}{A_0}$ for $((A \wedge B) \wedge C) \supset A_0$

and so on.

6.1 A First Shot

Some authors would be happy to endorse principles like the following:

t1 $\dfrac{p}{p \vee q\,\mathcal{B}\,p}$ $\qquad \dfrac{q}{p \vee q\,\mathcal{B}\,q}$

t2 $\dfrac{p \qquad q}{p \wedge q\,\mathcal{B}\,p, q}$

t3 $\dfrac{p}{\neg\neg p\,\mathcal{B}\,p}$

Yet they conflict with previously accepted postulates. By the previously accepted logic of factual equivalence, $p \approx p \vee p$, $p \approx p \wedge p$ and $p \approx \neg\neg p$ are validated. As a consequence, accepting any of these principles for $\mathcal{B}$ would result in a conflict with L-substitution and Irreflexivity** (replace q by p in **t1** and **t2**). Notice that the problem is intimately tied to the proposed logic of factual equivalence. On a conceptualist view of the sort depicted in the previous section, none of $p \approx p \vee p$, $p \approx p \wedge p$ and $p \approx \neg\neg p$ would be validated—actually, their negations would be. Conceptualists can, and arguably will, endorse **t1-t3**.

Conversely, I take it that anyone who accepts any one of these principles is committed to conceptualism.[16,17]

In order to avoid such a clash, one may wish to modify the principles by imposing suitable conditions:

T1 $\dfrac{p \vee q \not\approx p \qquad p}{p \vee q\, \mathcal{B}\, p} \qquad \dfrac{p \vee q \not\approx q \qquad q}{p \vee q\, \mathcal{B}\, q}$

T2 $\dfrac{p \wedge q \not\approx p \qquad p \wedge q \not\approx q \qquad p \qquad q}{p \wedge q\, \mathcal{B}\, p, q}$

T3 $\dfrac{\neg\neg p \not\approx p \qquad p}{\neg\neg p\, \mathcal{B}\, p}$

On the logic of factual equivalence I advocated, the second part of **T1** is redundant given the first part (and *vice versa*), and **T3** trivially holds because the negation of its first antecedent is validated. Yet given that $\neg\neg p \approx p$, is validated, by L-substitution and Irreflexivity**, $\neg\neg p\, \mathcal{B}\, p$ can never hold. We are thus left with the first part of **T1** (or its second part), and **T2**.

6.2 Some Problems, and Resulting Modifications

T1 and **T2** are problematic in the presence of a further plausible principle, namely:

N $\neg(p\, \mathcal{B} p \wedge q, \Delta)$

[16] Correia 2005, Fine msa, msb, Rosen 2010 and Schnieder ms explicitly endorse some of the three principles. (Fine msa does not deal with the logic of grounding and truth-functions, but he informally mentions **t2** as being correct.)

[17] At this point some words about Fine's notion of weak grounding are in order. Weak grounding is, just like grounding, many-to-one. Fine understands 'the fact that p is weakly grounded in the fact that q, and the fact that r, and ...' as:

(wg) For it to be the case that p is for it to be the case that q, and for it to be the case that r, and ...

The one-one locution 'for it to be the case that p is for it to be the case that q' is familiar enough, the many-to-one notion Fine has in mind is not. Crucially, (wg) is not to be understood as the one-one form 'for it to be the case that p is for it to be the case that (q and r and ...)', for Fine holds that although the fact that p is weakly grounded in the fact that p and the fact that p (granted that p is true), the fact that p can never be weakly grounded in the fact that $p \wedge p$. Fine cashes out the notion of weak grounding in terms of the notion of strict grounding as follows: $[p]$ is weakly grounded in $[q_1]$, $[q_2]$, ... iff (i) whenever $[p']$ is strictly grounded in $[p]$, $[r_1]$, $[r_2]$..., $[p']$ is strictly grounded in $[q_1]$, $[q_2]$, ..., $[r_1]$, $[r_2]$..., and (ii) whenever each of $[q_1]$, $[q_2]$, ... is strictly grounded in $[r_1]$, $[r_2]$..., $[p]$ is strictly grounded in $[r_1]$, $[r_2]$.... He mentioned to me that he would take factual equivalence to be just mutual weak ground ($p \approx q$ iff $[p]$ is weakly grounded in $[q]$, and $[q]$ in $[p]$). It might be thought that weak grounding can be defined as the converse of what I called disjunctive containment, more precisely that (wg) can be understood as $p \geqslant^d ((q \wedge r) \wedge ...)$. The two notions indeed share many formal properties. Yet Fine wants to say that for p true, the fact that $p \wedge p$ is weakly grounded in the fact that p, while $p \wedge p \geqslant^d p$ just means $\exists x(p \wedge p \approx p \vee x)$, and I do not think this is something a conceptualist like Fine would accept e.g. in case p is atomic. It is not clear to me whether weak grounding could be defined in terms of factual equivalence in some other way.

which, thanks to Permutation and Repetition, yields:

N* $\neg(p\,\mathcal{B}p \wedge q)$

(I will not take **N** as axiomatic since it follows from further axioms I will introduce in the next section.) Let me here argue informally. (1) Suppose p and $p \vee q \not\approx p$. Then by **T1**, $p \vee q\,\mathcal{B}p$. Suppose now that $p \geqslant^{cd} p \vee q$. Then for some x, $p \approx p \vee ((p \vee q) \wedge x)$. Then by the logic of factual equivalence, $p \approx (p \vee q) \wedge (p \vee x)$, and so by R-substitution, $p \vee q\,\mathcal{B}(p \vee q) \wedge (p \vee x)$—which contradicts **N***. (2) Suppose p, q, $p \wedge q \not\approx p$ and $p \wedge q \not\approx q$. Then by **T2** $p \wedge q\,\mathcal{B}p, q$. Suppose now that $p \geqslant^{cd} p \wedge q$. (The assumption that $q \geqslant^{cd} p \wedge q$ would yield to the same kind of problem.) Then for some x, $p \approx p \vee ((p \wedge q) \wedge x)$. Then by the logic of factual equivalence, $p \approx (p \vee (p \wedge q)) \wedge (p \vee x)$, and so by R-substitution, $p \wedge q\,\mathcal{B}(p \vee (p \wedge q)) \wedge (p \vee x), q$. Now either $p \wedge q \approx (p \vee (p \wedge q))$ or not. If not, then by **T1** and Cut, $p \vee (p \wedge q)\,\mathcal{B}(p \vee (p \wedge q)) \wedge (p \vee x), q$—which contradicts **N**. If so, then by L-substitution, $p \vee (p \wedge q)\,\mathcal{B}(p \vee (p \wedge q)) \wedge (p \vee x), q$—which again contradicts **N**.

(Notice that for a conceptualist, neither $p \geqslant^{cd} p \vee q$ nor $p \geqslant^{cd} p \wedge q$ can hold, so the previous difficulties do not arise for him.)

Clearly, one can escape these difficulties if instead of **T1** and **T2** one opts for:

TF1 $\dfrac{p \not\geqslant^{cd} p \vee q \qquad p}{p \vee q\,\mathcal{B}\,p}$ $\vee$*-introduction 1*

TF2 $\dfrac{p \not\geqslant^{cd} p \wedge q \qquad q \not\geqslant^{cd} p \wedge q \qquad p \qquad q}{p \wedge q\,\mathcal{B}\,p, q}$ $\wedge$*-introduction 1*

I shall take **TF1** and **TF2** as axiomatic. (There is no need to add:

- $\dfrac{q \not\geqslant^{cd} p \vee q \qquad q}{p \vee q\,\mathcal{B}\,q}$

since it follows from **TF1**, the logic of factual equivalence and L-substitution.)

6.3 Further Principles

TF1 is a sort of principle of disjunction-introduction for grounding which allows one to introduce disjunction from premises which do not contain $\mathcal{B}$, and **TF2** is a sort of principle of conjunction-introduction of the same type. I also take as axiomatic the following principles of introduction and elimination for both disjunction and conjunction which are of a different sort:

TF3 $\dfrac{p\,\mathcal{B}\,\Delta}{p \vee q\,\mathcal{B}\,\Delta}$ $\vee$*-introduction 2*

TF4 $\dfrac{p\,\mathcal{B}\,\Delta, (r \vee s) \qquad r}{p\,\mathcal{B}\,\Delta, r}$ $\vee$*-elimination*

TF5 $\dfrac{p\,\mathcal{B}\,\Delta \qquad r\,\mathcal{B}\,\Gamma}{p \wedge r\,\mathcal{B}\,\Delta, \Gamma}$ $\wedge$*-introduction 2*

TF6 $\dfrac{p \,\mathcal{B}\, \Delta, (r \wedge s)}{p \,\mathcal{B}\, \Delta, r, s}$ *∧-elimination*

Notice that thanks to Permutation and Repetition, **TF4** and **TF6** yield:

- $\dfrac{p \,\mathcal{B}\, (r \vee s) \qquad r}{p \,\mathcal{B}\, r}$
- $\dfrac{p \,\mathcal{B}\, (r \wedge s)}{p \,\mathcal{B}\, r, s}$

and that principle **N** from the previous subsection follows from **TF6** and Irreflexivity. Also notice that **TF3-TF6** follow from **t1**, **t2**, Cut and Permutation. So conceptualists who endorse the latter principles do not need **TF3-TF6** as extra principles.

6.4 The Reduction Axiom

Let us use $\Delta \not\geqslant^{cd} p$ for $p_1 \not\geqslant^{cd} p$ if Δ is p_1, for $(p_1 \not\geqslant^{cd} p \wedge p_2 \not\geqslant^{cd} p)$ if Δ is p_1, p_2, for $((p_1 \not\geqslant^{cd} p \wedge p_2 \not\geqslant^{cd} p) \wedge p_3 \not\geqslant^{cd} p)$ if Δ is p_1, p_2, p_3, etc. Informally speaking, $\Delta \not\geqslant^{cd} p$ just says that no item q in Δ is such that $q \geqslant^{cd} p$.

Given previously accepted principles, it can be shown that the following holds in G:

R$^+$ $(\widehat{\Delta} \wedge (p \geqslant^{d} \widehat{\Delta}) \wedge (\Delta \not\geqslant^{cd} p)) \supset (p \,\mathcal{B}\, \Delta)$

An informal proof runs as follows. (1) Suppose first that Δ is a list of one item, q. Suppose then $q \not\geqslant^{cd} p$. Then by the logic of factual equivalence, $q \not\geqslant^{cd} q \vee p$. Suppose now that q. By **TF1**, it follows that $q \vee p \,\mathcal{B}\, q$. Suppose now $p \geqslant^{d} q$. Then $p \approx q \vee p$, and so by L-substitution, we get $p \,\mathcal{B}\, q$. (2) Suppose now that Δ is a list of n items, q_1, ..., q_n, for $n \geqslant 2$. Suppose $q_1 \not\geqslant^{cd} p$, ..., $q_n \not\geqslant^{cd} p$. Then by the logic of factual equivalence, $q_1 \not\geqslant^{cd} q_1 \vee p$, ..., $q_n \not\geqslant^{cd} q_n \vee p$. Suppose now q_1, ..., q_n. By **TF1**, it follows that $q_1 \vee p \,\mathcal{B}\, q_1$, ..., $q_n \vee p \,\mathcal{B}\, q_n$. Define r_i, $1 \leqslant i \leqslant n$, recursively as follows: r_1 is $(q_1 \vee p)$; for every i such that $2 \leqslant i \leqslant n$, r_i is $(r_{i-1} \wedge (q_i \vee p))$. Then by **TF5**, we have $r_n \,\mathcal{B}\, \Delta$. Suppose now $p \geqslant^{d} \widehat{\Delta}$. Then $p \approx p \vee \widehat{\Delta}$, and so by the logic of factual equivalence, $p \approx r_n$. By L-substitution, we then get $p \,\mathcal{B}\, \Delta$.

We can also show that the following holds in G:

- $(p \,\mathcal{B}\, \Delta) \supset (\Delta \not\geqslant^{cd} p)$

An informal proof for the case where Δ has only one item, q, runs as follows. (The case where Δ has more than one item is exactly similar.) Suppose $p \,\mathcal{B}\, q$ and $q \geqslant^{cd} p$. Then for some x, $q \approx (p \wedge x) \vee q$. By the logic of factual equivalence, it follows that $q \approx (q \vee x) \wedge (p \vee q)$. By Substitution and **TF6**, we then get $p \,\mathcal{B}\, (q \vee x), (p \vee q)$. By Factivity, p, and so by **TF4** we get $p \,\mathcal{B}\, (q \vee r), p$, which violates Irreflexivity.

Since $(p \,\mathcal{B}\, \Delta) \supset \widehat{\Delta}$ directly follows from Factivity, we will thus have the converse of R$^+$, namely:

R$^-$ $(p\,\mathcal{B}\,\Delta) \supset (\hat{\Delta} \wedge (p \geqslant^d \hat{\Delta}) \wedge (\Delta \not\geqslant^{cd} p))$

provided that we accept the following postulate:

Re $(p\,\mathcal{B}\,\Delta) \supset (p \geqslant^d \hat{\Delta})$ *Reduction axiom*

I call it 'Reduction axiom' because in its presence, the following "reduction" principle follows:[18]

R $(p\,\mathcal{B}\,\Delta) \equiv (\hat{\Delta} \wedge (p \geqslant^d \hat{\Delta}) \wedge (\Delta \not\geqslant^{cd} p))$ *Reduction theorem*

That is to say, using the language of facts, a fact x is grounded in other facts iff (i) the latter facts all obtain, (ii) their conjunction is disjunctively contained in x, and (iii) none of these facts disjunctively contains some fact which conjunctively contains x. Still using factual talk, to say that $p \geqslant^d q$ is to say that the fact that p is a disjunctive fact with the fact that q as one of the disjuncts. The idea behind **Re** is thus that whenever a fact is grounded in a collection of facts, the grounded fact has a disjunctive nature, and the conjunction of the grounders is one of its disjuncts—as it were, grounding always arises via disjunction, in the way encoded by **TF1**.

Not all instances of **Re** are plausible. For instance, although one can maintain, with some plausibility, that the fact that {Socrates} exists is grounded in the fact that Socrates exists, view that the former fact is the disjunction of the latter fact and another fact is implausible. Thus, adding the Reduction axiom to the previous axiomatic basis yields a theory of grounding which at best takes care of certain types of grounding ties, and accordingly does not have full generality. Yet I will take the axiom to be part of G. The reason is pragmatic, so to speak: with the axiom on board, it is relatively easy to put forward a semantics for the system with respect to which the system is sound and complete (see next section), while without the axiom the task is significantly harder.

It would be highly desirable to have a semantical characterization of G minus **Re**. I hope to be able to provide one in the future. Yet, G is interesting in its own right, and accordingly having a semantical characterization for G is a good thing. It is also a good thing for another reason: an important consequence of the soundness of G that any (proper or improper) fragment of G—in particular G minus **Re**—is consistent.

7 A Semantics for G

In this section I formulate a semantics for language L and show that system G is sound and complete with respect to that semantics. The semantics is

[18] There is a striking formal similarity between the relationship between $\mathcal{B}$, $\geqslant^d$ and $\geqslant^{cd}$ as expressed by the Reduction theorem on one hand, and those between the Finean notions of strict grounding, weak grounding and partial weak grounding as Fine (msa and msb) sees them on the other hand. Fine indeed takes it that a fact is strictly grounded in other facts iff it is weakly grounded in them, and none of the latter facts is partly weakly grounded in the former fact. Yet, the similarity is of a limited character since, as we saw, weak grounding is not the converse of disjunctive containment, and accordingly partial weak grounding is not the converse of $\geqslant^{cd}$.

partly algebraic. It interprets the basic formulas as denoting what I will call 'facts', the truth-functional connectives as corresponding to certain operations on facts, factual equivalence as identity between facts, and grounding as a certain relation between facts. Thus the semantics can be seen as embodying what I earlier called a predicational view about grounding, and it actually can be modified in an obvious way to model languages for grounding which, unlike L, treat the notion as expressed by means of a predicate. Since it is my contention that the operational view on grounding is correct, I do not take the proposed formal semantics to have a non-ersatzist applied counterpart which provides truth-conditions for grounding claims.[19] In §7.1 I present system G in compact form, in §7.2 I present the semantics, and §7.3 deals with soundness and completeness.[20]

7.1 Review of the System

System G is given by all classical tautologies, a suitable basis for the sentential quantifier (any classical set of postulates will do), the rule Modus Ponens, and the following specific axioms:

Axioms for factual equivalence:

≈1 $p \approx \neg\neg p$
≈2 $p \approx p \wedge p$
≈3 $p \wedge q \approx q \wedge p$
≈4 $p \wedge (q \wedge r) \approx (p \wedge q) \wedge r$
≈5 $p \vee q \approx \neg(\neg p \wedge \neg q)$
≈6 $p \vee (q \wedge r) \approx (p \vee q) \wedge (p \vee r)$
≈7 $(p \approx q) \supset (\neg p \approx \neg q)$
≈8 $(p \approx q) \supset (p \wedge r \approx q \wedge r)$
≈9 $(p \approx q) \wedge (q \approx r) \supset (p \approx r)$
≈10 $(p \approx q) \supset (q \approx p)$
≈11 $(p \approx q) \supset (p \equiv q)$

Permutation and repetition:

P $(p \mathcal{B} \Delta) \supset (p \mathcal{B} \Gamma)$ — Γ a reordering of Δ *Permutation*
R $(p \mathcal{B} \Delta) \equiv (p \mathcal{B} \Delta, q)$ — q any item in Δ *Repetition*

Substitution principles:

S1 $(p \mathcal{B} \Delta) \wedge (q \approx p) \supset (q \mathcal{B} \Delta)$ *L-substitution*

[19] My stance towards the use of a "factualist" semantics is thus similar to that of the modal actualist who makes use of Kripke-style formal semantics for modal logics.

[20] Fine (msa and msb) also proposes a factualist semantics, but it is not algebraic. It is more similar to a regular model-theoretic semantics, although his basic semantic notion is not that of being true at a point (world, time, etc.), but that of being made true by facts. Schnieder ms proposes a semantics which is, as he himself grants, somewhat artificial, but which allows him to prove the consistency of his system.

S2 $(p\,\mathcal{B}\,\Delta, q) \wedge (q \approx r) \supset (p\,\mathcal{B}\,\Delta, r)$ *R-substitution*

Further structural principles:

G1 $(p\,\mathcal{B}\,\Delta) \supset (p \wedge \hat{\Delta})$ *Factivity*
G2 $\neg(p\,\mathcal{B}\,\Delta, p)$ *Irreflexivity*
G3 $(p\,\mathcal{B}\,\Delta, q) \wedge (q\,\mathcal{B}\,\Gamma) \supset (p\,\mathcal{B}\,\Delta, \Gamma)$ *Cut*

Principles for truth-functions:

TF1 $(p \not\geqslant^{cd} p \vee q) \wedge p \supset (p \vee q\,\mathcal{B}\,p)$ *$\vee$-introduction 1*
TF2 $(p \not\geqslant^{cd} p \wedge q) \wedge (q \not\geqslant^{cd} p \wedge q) \wedge p \wedge q \supset (p \wedge q\,\mathcal{B}\,p, q)$ *$\wedge$-introduction 1*
TF3 $(p\,\mathcal{B}\,\Delta) \supset (p \vee q\,\mathcal{B}\,\Delta)$ *$\vee$-introduction 2*
TF4 $(p\,\mathcal{B}\,\Delta, r \vee s) \wedge r \supset (p\,\mathcal{B}\,\Delta, r)$ *$\vee$-elimination*
TF5 $(p\,\mathcal{B}\,\Delta) \wedge (q\,\mathcal{B}\,\Gamma) \supset (p \wedge q\,\mathcal{B}\,\Delta, \Gamma)$ *$\wedge$-introduction 2*
TF6 $(p\,\mathcal{B}\,\Delta, r \wedge s) \supset (p\,\mathcal{B}\,\Delta, r, s)$ *$\wedge$-elimination*

Reduction principle:

Re $(p\,\mathcal{B}\,\Delta) \supset (p \geqslant^{d} \hat{\Delta})$ *Reduction axiom*

7.2 Semantics

We define a *factual structure* as a tuple $\langle F, \sqcap, -, ob\rangle$ where F (facts) is a non-empty set, $\sqcap$ (fact-product) and $-$ (fact-complementation) are binary / unary operations on facts meeting the following conditions (we define $\sqcup$ (fact-union) as follows: $\alpha \sqcup \beta := -(-\alpha \sqcap -\beta)$:

op1 $\alpha = --\alpha$
op2 $\alpha = \alpha \sqcap \alpha$
op3 $\alpha \sqcap \beta = \beta \sqcap \alpha$
op4 $\alpha \sqcap (\beta \sqcap \gamma) = (\alpha \sqcap \beta) \sqcap \gamma$
op5 $\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma)$
op6 $-\alpha \neq \alpha$

ob (obtainment) is a total function from F to $\{0, 1\}$ satisfying the following conditions:

ob1 $ob(\alpha \sqcap \beta) = 1$ iff $ob(\alpha) = 1$ and $ob(\beta) = 1$
ob2 $ob(-\alpha) = 1$ iff $ob(\alpha) = 0$

Then:

- $ob(\alpha \sqcup \beta) = 1$ iff $ob(\alpha) = 1$ or $ob(\beta) = 1$

We define the binary relations $\sqsupseteq^{d}$ and $\sqsupseteq^{cd}$ on F as follows: $\alpha \sqsupseteq^{d} \beta$ iff for some element u of F, $\alpha = (\beta \sqcup u)$; $\alpha \sqsupseteq^{cd} \beta$ iff for some element u of F, $\alpha \sqsupseteq^{d} (\beta \sqcap u)$. We have the following properties of $\sqsupseteq^{d}$ and $\sqsupseteq^{cd}$, structurally similar to those of $\geqslant^{d}$ and $\geqslant^{cd}$:

- $\alpha \sqsupseteq^d \alpha$
- If $\alpha \sqsupseteq^d \beta$ and $\beta \sqsupseteq^d \alpha$, then $\alpha = \beta$
- If $\alpha \sqsupseteq^d \beta$ and $\beta \sqsupseteq^d \gamma$, then $\alpha \sqsupseteq^d \gamma$
- If $\alpha \sqsupseteq^d \beta$, then $\alpha \wedge \gamma \sqsupseteq^d \beta \wedge \gamma$
- If $\alpha \sqsupseteq^d \beta$, then $\alpha \vee \gamma \sqsupseteq^d \beta \vee \gamma$
- If $\alpha \sqsupseteq^d \beta$, then $\alpha \vee \gamma \sqsupseteq^d \beta$
- If $\alpha \sqsupseteq^d \beta$, then $\alpha \sqsupseteq^d \alpha \wedge \beta$
- If $\alpha \sqsupseteq^d \beta$, then $\alpha \sqsupseteq^d \alpha \vee \beta$
- If $\alpha \sqsupseteq^d \beta$ and $ob(\beta)$, then $ob(\alpha)$

- $\alpha \sqsupseteq^{cd} \alpha$
- If $\alpha \sqsupseteq^{cd} \beta$ and $\beta \sqsupseteq^{cd} \gamma$, then $\alpha \sqsupseteq^{cd} \gamma$
- If $\alpha \sqsupseteq^{cd} \beta$, then $\alpha \wedge \gamma \sqsupseteq^{cd} \beta$
- If $\alpha \sqsupseteq^{cd} \beta \wedge \gamma$, then $\alpha \sqsupseteq^{cd} \beta$
- If $\alpha \sqsupseteq^{cd} \beta \vee \gamma$, then $\alpha \sqsupseteq^{cd} \beta$
- If $\alpha \sqsupseteq^{cd} \beta$, then $\alpha \vee \gamma \sqsupseteq^{cd} \beta$

- If $\alpha \sqsupseteq^d \beta$, then $\alpha \sqsupseteq^{cd} \beta$
- If $\alpha \sqsupseteq^d \beta$ and $\beta \sqsupseteq^{cd} \gamma$, then $\alpha \sqsupseteq^{cd} \gamma$
- If $\alpha \sqsupseteq^{cd} \beta$ and $\beta \sqsupseteq^d \gamma$, then $\alpha \sqsupseteq^{cd} \gamma$

- $\alpha \sqsupseteq^d \beta$ iff $\alpha = \beta \sqcup \alpha$
- $\alpha \sqsupseteq^{cd} \beta$ iff $\alpha \sqsupseteq^d \beta \sqcap \alpha$

There are factual structures. Take for instance any non-empty set S and let s be one of its members. We can then define a factual structure where F is the power set of S, $\sqcap$ is set-intersection, $-$ is set-complementation, and *ob* is defined as follows: for every $S' \in F$, $ob(S') = 1$ in case $s \in S'$, $ob(S') = 0$ otherwise.

A *factual model* is a tuple $\langle F, \sqcap, -, ob, [\cdot]\rangle$, where $\langle F, \sqcap, -, ob\rangle$ is a factual structure and $[\cdot]$ an *interpretation*, i.e. a function which takes every sentential constant of the language into a fact in F. Relative to a factual model, an *assignment* is a function which takes every sentential variable of the language into a fact of the model. Two assignments are x-*alternatives* iff they differ at most on the value they assign to variable x.

Given any factual model $M = \langle F, \sqcap, -, ob, [\cdot]\rangle$ and any assignment μ relative to M, we define the function $[\cdot]_\mu$ over the atoms as follows: $[a]_\mu := [a]$; $[x]_\mu := \mu(x)$. Function $[\cdot]_\mu$ is then extended to arbitrary basic sentences thanks to the following clauses:

- $[p \wedge q]_\mu = [p]_\mu \sqcap [q]_\mu$
- $[\neg p]_\mu = -[p]_\mu$
- $[p \vee q]_\mu = [p]_\mu \sqcup [q]_\mu$

We then have:

o1 $ob([p \wedge q]_\mu) = 1$ iff $ob([p]_\mu) = 1$ and $ob([q]_\mu) = 1$

o2 $ob([\neg p]_\mu) = 1$ iff $ob([p]_\mu) = 0$
o3 $ob([p \vee q]_\mu) = 1$ iff $ob([p]_\mu) = 1$ or $ob([q]_\mu) = 1$

Given a factual model $M = \langle F, \sqcap, -, ob, [\cdot] \rangle$, we define truth in M with respect to an assignment in the obvious way:

1. $M \vDash_\mu p$ iff $ob([p]_\mu) = 1$
2. $M \vDash_\mu p \approx q$ iff $[p]_\mu = [q]_\mu$
3. $M \vDash_\mu p\mathcal{B}q_1...q_n$ iff
   - For all i such that $1 \leqslant i \leqslant n$, $ob([q_i]_\mu) = 1$
   - $[p]_\mu \sqsupseteq^d [q_1]_\mu \sqcap ... \sqcap [q_n]_\mu$
   - For all i such that $1 \leqslant i \leqslant n$, not: $[q_i]_\mu \sqsupseteq^{cd} [p]_\mu$
4. $M \vDash_\mu A \wedge B$ iff $M \vDash_\mu A$ and $M \vDash_\mu B$
5. $M \vDash_\mu \neg A$ iff $M \nvDash_\mu A$
6. $M \vDash_\mu A \vee B$ iff $M \vDash_\mu A$ or $M \vDash_\mu B$
7. $M \vDash_\mu \exists x A$ iff for some x-alternative ρ of μ, $M \vDash_\rho A$

(Given o1-o3, the presence of 4-6 alongside with 1 does not yield trouble.) We have the following derived clauses:

- $M \vDash_\mu p \geqslant^d q$ iff $[p]_\mu \sqsupseteq^d [q]_\mu$
- $M \vDash_\mu p \geqslant^{cd} q$ iff $[p]_\mu \sqsupseteq^{cd} [q]_\mu$

A formula is finally said to be *valid* iff it is true in every model with respect to every assignment.

7.3 Soundness and Completeness

It is not difficult, although it takes some patience, to show that system G is sound respect to the proposed semantics, i.e. that:

Soundness. Every theorem of G is valid.

Given the existence of factual models, this establishes that G is consistent, as well as any part of it—in particular the system obtained by dropping the controversial axiom schema **Re**.

For completeness, we use a standard Henkin-style method. Let Σ be a consistent set of formulas of L. Let then L^+ be any language which differs from L only in that it has infinitely more sentential constants than L. By a classical result, there exists a set S of L^+-formulas which contains Σ, is maximal, consistent and $\exists$-complete. We build a model based on S and define an assignment relative to that model, and show that every formula of S is true, relative to that assignment, in that model. From now on, we shall work exclusively with L^+-formulas.

We have, classically:

- $A \wedge B \in S$ iff $A \in S$ and $B \in S$

- $\neg A \in S$ iff $A \notin S$
- $A \vee B \in S$ iff $A \in S$ or $B \in S$
- $\exists x A \in S$ iff for some sentential constant a, $A[a/x] \in S$

We define the relation $\sim$ on the basic formulas of the language as follows: $p \sim q$ iff $p \approx q \in S$. $\sim$ is an equivalence relation and we let F be the set of all the corresponding equivalence classes. We denote the equivalence class of basic formula p by '$\overline{p}$'. Notice that:

- For every $s \in F$, if for some $p \in s$, $p \in S$, then for every $q \in s$, $q \in S$

We define the operations $\sqcap$ and $-$ on F as follows:

- $p \in s \sqcap t$ iff for some $q_1 \in s$, $q_2 \in t$, $p \approx q_1 \wedge q_2 \in S$
- $p \in -s$ iff for some $q \in s$, $p \approx \neg q \in S$

We then have:

- $p \in s \sqcup t$ iff for some $q_1 \in s$, $q_2 \in t$, $p \approx q_1 \vee q_2 \in S$

We can show that:

- For all $s, t \in F$, if for some $q_1 \in s$, $q_2 \in t$, $p \approx q_1 \wedge q_2 \in S$, then for all $q_3 \in s$, $q_4 \in t$, $p \approx q_3 \wedge q_4 \in S$
- For every $s \in F$, if for some $q \in s$, $p \approx \neg q \in S$, then for all $r \in s$, $p \approx \neg r \in S$
- For all $s, t \in F$, if for some $q_1 \in s$, $q_2 \in t$, $p \approx q_1 \vee q_2 \in S$, then for all $q_3 \in s$, $q_4 \in t$, $p \approx q_3 \vee q_4 \in S$

and:

- $\overline{p \wedge q} = \overline{p} \sqcap \overline{q}$
- $\overline{\neg p} = -\overline{p}$
- $\overline{p \vee q} = \overline{p} \sqcup \overline{q}$

It is easy to prove that the operations meet conditions op1-op6 above. We define *ob* by: $ob(s) = 1$ if for some $p \in s$, $p \in S$, and $ob(s) = 0$ otherwise. Then *ob* meets the conditions ob1 and ob2. We have then established that $\langle F, \sqcap, -, ob \rangle$ is a factual structure.

We finally define $[\cdot]$ by putting $[a] = \overline{a}$, and μ by putting $\mu(x) = \overline{x}$. We can then show (by induction on the length of the formulas) that for every basic formula p, $[p]_\mu = \overline{p}$. We can also show:

- $ob([p]_\mu) = 1$ iff $p \in S$
- $[p]_\mu = [q]_\mu$ iff $p \approx q \in S$
- $[p]_\mu \sqsupseteq^d [q]_\mu$ iff $p \geqslant^d q \in S$
- $[p]_\mu \sqsupseteq^{cd} [q]_\mu$ iff $p \geqslant^{cd} q \in S$

Thanks to result, it is easy to establish:

Truth-lemma. For every L^+-formula A, A is true in model $\langle F, \sqcap, -, ob, [\cdot] \rangle$ relative to μ iff $A \in S$.

The proof is by induction on the complexity of the formulas, and the case of formulas of type $p\,\mathcal{B}\,\Delta$ is dealt with indirectly, by using the Reduction theorem (see §6.4).

It then follows that all the formulas in S, and so all those in Σ, are true in the constructed model. Given that Σ was an arbitrary consistent set of formulas, we have:

Strong completeness. Every set of L-formulas which is consistent (relative to G) is satisfiable, i.e. has all its members true in some model relative to some assignment.

And so:

Weak completeness. Every valid formula is a theorem of G.

Remember that by d1 and d2 (§5), $\geqslant^d$ and $\geqslant^{cd}$ are definable without the help of $\exists$. Let then L_P be a purely propositional counterpart of L, i.e. a language just like L but without sentential variables and the quantifier. Then define $p \geqslant^d q$ as $p \approx q \vee p$ and $p \geqslant^{cd} q$ as $p \geqslant^d q \wedge p$, and consider the system G_P defined on L_P exactly like G but without the postulates for existential quantification. Finally modify the semantics for L in the obvious way so as to obtain a semantics for L_P. It should then be clear that G_P is sound and complete with respect to the modified semantics.[21]

References

Anderson, A. R., and N. D. Belnap. 1975. *Entailment: The Logic of Relevance and Necessity*, vol I, Princeton University Press.

Angell, R. B. 1977. "Three Systems of First Degree Entailment", *Journal of Symbolic Logic*, 42:147.

Angell, R. B. 1989. "Deducibility, Entailment and Analytic Containement", in J. Norman and R. Sylvan (eds.), *Directions in Relevant Logic*, Dordrecht: Kluwer Academic Publishers.

Audi, P. ms. "Grounding", *Manuscript*.

Bolzano, B. 1973. *Theory of Science*, Dordrecht: D. Reidel.

Correia, F. 2004. "Semantics for Analytic Containment", *Studia Logica*, 77: 87–104.

[21] In Correia 2004 I develop a semantics for Angell's logic of analytic containment (analytic containment is definable in terms of analytic equivalence, and *vice versa*) which could easily be used to model G_P. It would be interesting to see how the semantics, which does not handle quantification, could be modified in order to model G.

Correia, F. 2005. *Existential Dependence and Cognate Notions*, Munich: Philosophia.

Fine, K. 2001. "The Question of Realism", *Philosopher's Imprint*, 1(1).

Fine, K. msa. "The Pure Logic of Ground", *Manuscript.*

Fine, K. msb. "A Guide to Ground", *Manuscript.*

Mulligan, K. 2004. "Essence and Modality. The Quintessence of Husserl's Theory", in M. Siebel and M. Textor (eds.) *Semantik und Ontologie. Beitrge zur philosophischen Forschung*, Frankfurt: Ontos Verlag, 387–418.

Rosen, G. 2010. "Metaphysical Dependence: Grounding and Reduction", in B. Hale and A. Hoffmann (eds.), *Modality: Metaphysics, Logic, and Epistemology*, Oxford University Press.

Schnieder, B. ms. "A Logic of 'Because' ", *Manuscript.*

Schnieder, B. 2004. "The Ability to Render Something False", *Proceedings of the Aristotelian Society*, 104, 295–303.

Schnieder, B. 2006a. "A Certain Kind of Trinity: Dependence, Substance, Explanation", *Philosophical Studies*, 129: 393–419.

Schnieder, B. 2006b. "Truth-Making without Truth-Makers", *Synthese*, 152: 21–46.

Schnieder, B. 2006c. "Troubles with Truth-making: Necessitation and Projection", *Erkenntnis*, 64: 61–74.

Schnieder, B. 2008. "Truth-functionality", *Review of Symbolic Logic*, 1: 64–72.